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Letter to the Editor

A condition for the nonsymmetric saddle point matrix being diagonalizable and having real and positive eigenvalues

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Abstract

This paper discusses the spectral properties of the nonsymmetric saddle point matrices of the form $\mathscr{A} = [A\ \ B^{\mathrm{T}};\ -B\ \ C]$ with A symmetric positive definite, B full rank, and C symmetric positive semidefinite. A new sufficient condition is obtained so that $\mathscr{A}$ is diagonalizable with all its eigenvalues real and positive. This condition is weaker than that stated in the recent paper [J. Liesen, A note on the eigenvalues of saddle point matrices, Technical Report 10-2006, Institute of Mathematics, TU Berlin, 2006].

MSC: 65F10

Keywords: Saddle point matrix; Eigenvalue; Spectral condition number

1. Introduction

In this paper, we investigate the spectral properties of the nonsymmetric saddle point matrices of the following form:

$$\mathscr{A} = \begin{bmatrix} A & B^{\mathrm{T}} \\ -B & C \end{bmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite ($A \succ 0$), $B \in \mathbb{R}^{m\times n}$ has full rank with $m \leqslant n$, and $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite ($C \succeq 0$). Matrices such as (1) can arise, for example, from finite element discretizations of Stokes equations and Maxwell equations, nonlinearly constrained optimizations, fluid dynamics and incompressible elasticity. See Benzi et al. [2] for a comprehensive survey.

If certain conditions are satisfied so that $\mathscr{A}$ is diagonalizable, and has real positive eigenvalues, then it is advantageous to analyze the convergence of Krylov subspace methods for solving the linear systems with $\mathscr{A}$; see Benzi and Simoncini [4]. Moreover, this gives rise to a three-term recurrence conjugate gradient type method based on a positive definite inner product; see [7,8]. Some researchers have been devoted to deriving sufficient conditions for $\mathscr{A}$ being diagonalizable

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and having real and positive eigenvalues. Fischer et al. [5] first studied $\mathscr{A}$ with $A = \eta I_n \succ 0$ and $C = 0$. The results in [5] have been extended to $\mathscr{A}$ with $A \succ 0$ and $C = 0$ in [4], and further generalized to $\mathscr{A}$ with $A \succ 0$ and $C \succeq 0$ in [7].

Let A, B and C be defined in (1). Then denote

$$\mu_1 = \lambda_{\min}(A), \quad \mu_n = \lambda_{\max}(A), \quad \delta_1 = \lambda_{\min}(C),$$

$$\delta_m = \lambda_{\max}(C) \quad \text{and} \quad \sigma_m = \lambda_{\max}(BA^{-1}B^{\mathrm{T}}).$$

It is shown in Liesen [7] that $\mathscr{A}$ is diagonalizable, and has real and positive eigenvalues if

$$\mu_1 > 4(\delta_m + \sigma_m). \tag{2}$$

It seems that the condition above is bit too restrictive when the maximum eigenvalue of C, i.e., δ_m, is large. This paper is devoted to giving a new sufficient condition which is weaker than (2).

2. Main results

Let a matrix $\mathscr{A}$ be defined in (1) with $A \succ 0$, B full rank, and $C \succeq 0$. Then we define the symmetric matrix

$$\mathscr{G}_C(\gamma) = \begin{bmatrix} A - \gamma I_n & B^{\mathrm{T}} \\ B & \gamma I_m - C \end{bmatrix},$$

where γ is a yet to be specified real scalar. It is immediate to verify that $\mathscr{G}_C(\gamma)\mathscr{A} = \mathscr{A}^{\mathrm{T}}\mathscr{G}_C(\gamma)$. It follows from [7,4] that $\mathscr{G}_C(\gamma) \succ 0$ *is a sufficient condition so that* $\mathscr{A}$ *is diagonalizable with all its eigenvalues real and positive.*

We now give the following main result.

Theorem 2.1. *Let* $A \succ 0$, $C \succeq 0$, B *full rank*, $\tilde{\gamma} = \frac{1}{2}(\mu_1 + \delta_m)$ *and*

$$\mu_1 > \delta_m + 2\sqrt{\mu_1 \sigma_m}. \tag{3}$$

Then $\mathscr{G}_C(\tilde{\gamma})$ *is positive definite, and* $\mathscr{A}$ *is diagonalizable with all its eigenvalues real and positive. Moreover, the spectral condition number of* $\mathscr{G}_C(\tilde{\gamma})$ *satisfies*

$$\kappa(\mathscr{G}_C(\tilde{\gamma})) := \frac{\lambda_{\max}(\mathscr{G}_C(\tilde{\gamma}))}{\lambda_{\min}(\mathscr{G}_C(\tilde{\gamma}))} < \frac{2(\mu_n - \delta_1 + \delta_m)}{\mu_1 - \delta_m - 2\sqrt{\mu_1 \sigma_m}}. \tag{4}$$

Proof. Let θ be any eigenvalue of $\mathscr{G}_C(\tilde{\gamma})$, and $[x^*, y^*]^*$ be the corresponding eigenvector. Here, x^* denotes the conjugate transpose of vector x. Then $\mathscr{G}_C(\tilde{\gamma})[x^*, y^*]^* = \theta[x^*, y^*]^*$, i.e.,

$$Ax - \tilde{\gamma}x + B^{\mathrm{T}}y = \theta x, \tag{5}$$

$$Bx + \tilde{\gamma}y - Cy = \theta y. \tag{6}$$

Consider now the following three cases:

Case (i): $(\theta - \tilde{\gamma})I_m + C$ is singular, or indefinite. Clearly, $\tilde{\gamma} - \delta_m \leqslant \theta \leqslant \tilde{\gamma} - \delta_1$.

Case (ii): $(\theta - \tilde{\gamma})I_m + C \succ 0$. It is easy to see that $\theta > \tilde{\gamma} - \delta_1$. Obtaining $y = ((\theta - \tilde{\gamma})I_m + C)^{-1}Bx$ from (6), and substituting y into (5) yields

$$Ax - \tilde{\gamma}x + B^{\mathrm{T}}((\theta - \tilde{\gamma})I_m + C)^{-1}Bx = \theta x$$

and then, after multiplying from the left with x^*,

$$x^*Ax - \tilde{\gamma}x^*x + x^*B^{\mathrm{T}}((\theta - \tilde{\gamma})I_m + C)^{-1}Bx = \theta x^*x. \tag{7}$$

Note that it must be $x \neq 0$ for if otherwise (6) would imply $y = 0$, which contradicts that $[x^*, y^*]^*$ is an eigenvector. Since

$$(\theta - \tilde{\gamma})I_m + C \succeq (\theta - \tilde{\gamma} + \delta_1)I_m \succ 0,$$

it follows from (7) that

$$x^*Ax - \tilde{\gamma}x^*x + \frac{1}{\theta - \tilde{\gamma} + \delta_1}x^*B^{\mathrm{T}}Bx \geqslant \theta x^*x,$$

which, from simple manipulations, can be rewritten as

$$\theta - \tilde{\gamma} + \delta_1 + \frac{x^*B^{\mathrm{T}}Bx}{x^*Ax} \geqslant (\theta - \tilde{\gamma} + \delta_1)(\theta + \tilde{\gamma})\frac{x^*x}{x^*Ax}. \tag{8}$$

Bounding the left-hand side of (8) from above by

$$\theta - \tilde{\gamma} + \delta_1 + \frac{x^*B^{\mathrm{T}}Bx}{x^*Ax} \leqslant \theta - \tilde{\gamma} + \delta_1 + \sigma_m,$$

and the right-hand side of (8) from below by

$$(\theta - \tilde{\gamma} + \delta_1)(\theta + \tilde{\gamma})\frac{x^*x}{x^*Ax} \geqslant (\theta - \tilde{\gamma} + \delta_1)(\theta + \tilde{\gamma})\frac{1}{\mu_n}$$

yields

$$\theta - \tilde{\gamma} + \delta_1 + \sigma_m \geqslant (\theta - \tilde{\gamma} + \delta_1)(\theta + \tilde{\gamma})\frac{1}{\mu_n},$$

which leads to the following quadratic inequality in θ:

$$\theta^2 - (\mu_n - \delta_1)\theta - \tilde{\gamma}^2 + (\mu_n + \delta_1)\tilde{\gamma} - (\delta_1 + \sigma_m)\mu_n \leqslant 0. \tag{9}$$

Solving (9) we obtain the upper bound of θ:

$$\begin{aligned}\theta &\leqslant \tfrac{1}{2}\left(\mu_n - \delta_1 + \sqrt{(\mu_n - \delta_1)^2 + 4(\tilde{\gamma}^2 - (\mu_n + \delta_1)\tilde{\gamma} + (\delta_1 + \sigma_m)\mu_n)}\right)\\ &= \tfrac{1}{2}\left(\mu_n - \delta_1 + \sqrt{(\mu_1 - \mu_n - \delta_1 + \delta_m)^2 + 4\mu_n\sigma_m}\right).\end{aligned}$$

Case (iii): $(\theta - \tilde{\gamma})I_m + C \prec 0$. In this case, it is obvious that $\theta < \tilde{\gamma} - \delta_m$. Because of $x \neq 0$ from the proof in Case (ii), and

$$(\theta - \tilde{\gamma})I_m + C \preceq (\theta - \tilde{\gamma} + \delta_m)I_m \prec 0,$$

using (7) we can get

$$x^*Ax - \tilde{\gamma}x^*x + \frac{1}{\theta - \tilde{\gamma} + \delta_m}x^*B^{\mathrm{T}}Bx \leqslant \theta x^*x,$$

which, by simple manipulations, is equivalent to

$$\theta - \tilde{\gamma} + \delta_m + \frac{x^*B^{\mathrm{T}}Bx}{x^*Ax} \geqslant (\theta + \tilde{\gamma})(\theta - \tilde{\gamma} + \delta_m)\frac{x^*x}{x^*Ax}. \tag{10}$$

It follows from (3) that $2\tilde{\gamma} - \delta_m - \sigma_m > 0$, and then $2\tilde{\gamma} - C - BA^{-1}B^{\mathrm{T}} \succ 0$, which, together with $A \succ 0$ and [6], leads to

$$\mathscr{B} := \mathscr{G}_C(\tilde{\gamma}) + \tilde{\gamma}I_{n+m} \succ 0.$$

It is clear that $\theta + \tilde{\gamma} > 0$, since $\theta + \tilde{\gamma}$ is the eigenvalue of $\mathscr{B}$. Thus, bounding the right-hand side of (10) from below by $(\theta + \tilde{\gamma})(\theta - \tilde{\gamma} + \delta_m)\mu_1^{-1}$, and the left-hand side of (10) from above by $\theta - \tilde{\gamma} + \delta_m + \sigma_m$ yields

$$\theta - \tilde{\gamma} + \delta_m + \sigma_m \geqslant (\theta + \tilde{\gamma})(\theta - \tilde{\gamma} + \delta_m)\frac{1}{\mu_1},$$

which can be rewritten as the quadratic inequality in θ:

$$\theta^2 - (\mu_1 - \delta_m)\theta - \tilde{\gamma}^2 + (\mu_1 + \delta_m)\tilde{\gamma} - (\delta_m + \sigma_m)\mu_1 \leqslant 0. \tag{11}$$

Solving the inequality (11) above, we derive the lower bound of θ:

$$\begin{aligned}\theta &\geqslant \tfrac{1}{2}(\mu_1 - \delta_m - \sqrt{(\mu_1 - \delta_m)^2 + 4(\tilde{\gamma}^2 - (\mu_1 + \delta_m)\tilde{\gamma} + (\delta_m + \sigma_m)\mu_1)}) \\ &= \tfrac{1}{2}(\mu_1 - \delta_m - 2\sqrt{\mu_1\sigma_m}).\end{aligned}$$

Due to Cases (i)–(iii) and the assumption (3), it deduces

$$\theta \geqslant \min\{\tilde{\gamma} - \delta_m, \tilde{\gamma} - \delta_1, \tfrac{1}{2}(\mu_1 - \delta_m - 2\sqrt{\mu_1\sigma_m})\} = \tfrac{1}{2}(\mu_1 - \delta_m - 2\sqrt{\mu_1\sigma_m}) > 0, \tag{12}$$

which implies that $\mathscr{G}_C(\tilde{\gamma})$ is positive definite. Thus, $\mathscr{A}$ is diagonalizable with all its eigenvalues real and positive. On the other hand, it follows from Cases (i)–(iii) that

$$\begin{aligned}\theta &\leqslant \max\{\tilde{\gamma} - \delta_1, \tfrac{1}{2}(\mu_n - \delta_1 + \sqrt{(\mu_1 - \mu_n - \delta_1 + \delta_m)^2 + 4\mu_n\sigma_m}), \tilde{\gamma} - \delta_m\} \\ &= \tfrac{1}{2}(\mu_n - \delta_1 + \sqrt{(\mu_1 - \mu_n - \delta_1 + \delta_m)^2 + 4\mu_n\sigma_m}).\end{aligned} \tag{13}$$

From (3) we obtain $\mu_1 > 4\sigma_m$, and then

$$\begin{aligned}&\mu_n - \delta_1 + \delta_m - \tfrac{1}{2}(\mu_n - \delta_1 + \sqrt{(\mu_1 - \mu_n - \delta_1 + \delta_m)^2 + 4\mu_n\sigma_m}) \\ &\quad > \mu_n - \delta_1 + \delta_m - \tfrac{1}{2}(\mu_n - \delta_1 + \sqrt{(\mu_1 - \mu_n - \delta_1 + \delta_m)^2 + \mu_n\mu_1}) \\ &\quad = \frac{(\mu_1 + 2\delta_m)(\mu_n - \mu_1) + 4\mu_n(\delta_m - \delta_1) + 2\delta_m(\delta_m - \delta_1) + \delta_m^2 + 2\mu_1\delta_1}{2(\mu_n + 2\delta_m - \delta_1 + \sqrt{(\mu_1 - \mu_n - \delta_1 + \delta_m)^2 + \mu_n\mu_1})} \geqslant 0.\end{aligned} \tag{14}$$

Combining (12)–(14) yields the upper bound of $\kappa(\mathscr{G}_C(\tilde{\gamma}))$. This completes the proof. □

Some remarks on Theorem 2.1 are given as follows:

- Since the sufficient condition (3) can be rewritten as

$$\mu_1 > \delta_m + 2\sigma_m + 2\sqrt{\sigma_m^2 + \delta_m\sigma_m}, \tag{15}$$

it is elementary to find that the condition (3) is weaker than (2) derived in [7].
- For the important case $C = \beta I_m \succeq 0$ (see e.g., [1,3]), [4, Corollary 2.6] shows that if

$$\mu_1 > 3\beta + 4\sigma_m, \tag{16}$$

then all eigenvalues of $\mathscr{A}$ are real. In fact, if (16) is satisfied, by (3) or (15) we get that $\mathscr{A}$ not only has real and positive eigenvalues, but also is diagonalizable.
- The upper bounds of the spectral condition number of $\mathscr{G}_C(\tilde{\gamma})$ can be used to estimate the convergence rate of the (nonstandard) conjugate gradient iteration; see [7,4]. For the case $C = \beta I_m \succeq 0$, the upper bound (4) becomes

$$\kappa(\mathscr{G}_{\beta I_m}(\tilde{\gamma})) < \frac{2\mu_n}{\mu_1 - \beta - 2\sqrt{\mu_1\sigma_m}}.$$

For the case $C = 0$, the upper bound above reduces to

$$\kappa(\mathscr{G}_0(\tilde{\gamma})) < \frac{2\mu_n}{\mu_1 - 2\sqrt{\mu_1\sigma_m}},$$

where $\tilde{\gamma} = \frac{1}{2}\mu_1$, and $\mathscr{G}_0(\tilde{\gamma})$ is equal to the matrix G defined in [4, p. 182]. It is clear that the upper bound above is more accurate than that given in [4, Corollary 3.2] because of

$$\frac{1}{2}(\mu_1 - 2\sqrt{\mu_1\sigma_m}) > \frac{1}{4}\mu_1 - \sigma_m.$$

3. An example

Consider the following matrix given in [7]:

$$\mathscr{A} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & b & 0 \\ 0 & 2 & 0 & 0 & b \\ 0 & 0 & 3 & 0 & 0 \\ \hline -b & 0 & 0 & 2c & -c \\ 0 & -b & 0 & -c & 2c \end{array}\right) \in \mathbb{R}^{5\times 5}, \ \ b \neq 0, \ \ c \geqslant 0.$$

It is immediate to obtain $\mu_1 = 1$, $\sigma_2 = b^2$, $\delta_2 = 3c$. Thus, the sufficient condition (2) reduces to

$$1 > 12c + 4b^2, \tag{17}$$

and the sufficient condition (3) becomes

$$1 > 3c + 2|b|. \tag{18}$$

If c is chosen to be $\frac{1}{12}$, the sufficient condition (17) is not satisfied whatever b is. From (18) we find that $\mathscr{A}$ is diagonalizable with all its eigenvalues real and positive whenever $|b| < \frac{1}{2}(1-3c) = 0.3750$. In fact, $\mathscr{A}$ has five distinct real and positive eigenvalues whenever $|b| < 0.4056855$ by a MATLAB computation; see [7].

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